MULTIPLE SOLUTIONS FOR ELLIPTIC EQUATIONS WITH DIFFERENT KINDS OF NONSTANDARD GROWTH CONDITIONS

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Abstract

In this paper, we investigate the existence and multiplicity of solutions for \( p(x) \)-Laplacian equations with different kinds of boundary conditions by using the Ricceri’s variational principle. Sufficient conditions are obtained for the existence of at least three solutions of the equation under consideration and our result generalizes the corresponding known results.

1. Introduction

In this paper, we will consider the following \( p(x) \)-Laplacian equations with different kinds of boundary conditions:

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\[
\begin{cases}
- \text{div}(|u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\
Bu = 0 & \text{on } \partial \Omega,
\end{cases}
\] (P)

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary, \( \lambda > 0, \mu \geq 0 \) are real numbers, \( p(x) \in C(\overline{\Omega}) \) with \( \inf_{x \in \Omega} p(x) > N \). The main interest in studying the \( p(x) \)-Laplace operator \( -\text{div}(|u|^{p(x)-2} \nabla u) \) of such problems arises from the study of electrorheological fluids and elastic mechanics. The \( p(x) \)-Laplace operator \( -\text{div}(|u|^{p(x)-2} \nabla u) \) is a generalization of the classical \( p \)-Laplace operator \( -\text{div}(|u|^{p-2} \nabla u) \).

We will consider the case of the boundary conditions as follows:

1. \( B = B_1 \), Neumann boundary condition, i.e.,
\[
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\]
where the \( \nu \) is the outward unit normal to \( \partial \Omega \);

2. \( B = B_2 \), Dirichlet boundary condition, i.e.,
\[
u \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega.
\]

In this paper, we assume \( f(x, u) \) and \( g(x, u) \) satisfy the following conditions:

\( (f_1) \) \( f, g : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Caratheodory function and there exist two positive numbers \( c_1, c_2 \) such that
\[
|g(x, t)|, |f(x, t)| \leq c_1 + c_2 |t|^{\alpha(x)-1}, \quad \forall (x, t) \in \Omega \times \mathbb{R},
\]
where \( \alpha \in C(\overline{\Omega}), 1 < \alpha^- = \min_{x \in \Omega} \alpha(x) < \alpha^+ = \max_{x \in \Omega} \alpha(x) < p^- = \min_{x \in \Omega} p(x) \).

\( (f_2) \) \( f(x, t) < 0, \quad \text{when } |t| \in (0, 1); \)
\[
f(x, t) \geq M > 0, \quad \text{when } |t| \in (t_0, \infty) \quad \exists t_0 > 1.
\]

We will study Equation (P) in the case, when \( p(x) > N \) for any \( x \in \overline{\Omega} \), which exists at least three weak solutions in a variable exponent Sobolev
space by using a three critical point theorem due to Ricceri (see Theorem 1 in [19]).

In recent years, many publications [1-4, 12, 14, 15] have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications. Existence and multiplicity results for quasilinear elliptic systems with variational structure have been broadly investigated.

In [12], Fan and Han ensured the existence and multiplicity of solutions for the problem

\[
\begin{cases}
- \text{div}(|\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = f(x, u) \quad \text{in } \mathbb{R}^N, \\
u \in W^{1, p(x)}(\mathbb{R}^N),
\end{cases}
\]

where \( N \geq 2, \ p(x) \) is a function defined on \( \mathbb{R}^N \).

The problem (P) was investigated by Liu [14]. The differences between these two papers are the assumptions in this paper are simpler than that in [14], and the use of a new tool made the proof very simple.

In [15], Mihailescu studied the particular case

\[
f(x, t) = |t|^{q(x)-2} t - t, \quad g(x, t) = 0,
\]

where \( q(x) \in C_+(\Omega) \) satisfies \( 2 < q(x) < \inf_{y \in (\overline{\Omega})} p(y) \) for any \( x \in \overline{\Omega} \). He established the existence of at least three weak solutions by using the Ricceri’s variational principle for \( p(x) > N \).

The paper is organized as follows. We will introduce some basic proposition and preliminary results in Section 2, including the variable exponent Lebesgue, Sobolev spaces and Ricceri’s three-critical-points theorem. In part 3, we will give the main result and its proof.

2. Preliminaries

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N (N \geq 1) \) with a smooth boundary \( \partial \Omega \). Set
\[ C_\ast(\overline{\Omega}) = \{ h \mid h \in C(\overline{\Omega}), \quad h(x) > 1 \text{ for all } x \in \overline{\Omega} \}. \]

\[ L^\infty_\ast(\Omega) = \{ p \in L^\infty(\Omega) : \text{ess inf}_{x \in \Omega} p(x) > 1 \}. \]

For \( p \in L^\infty_\ast(\Omega) \), denote

\[ p^- = p^- (\Omega) = \text{ess inf}_{x \in \Omega} p(x), \quad p^+ = p^+ (\Omega) = \text{ess sup}_{x \in \Omega} p(x). \]

Define

\[ L^p(x)(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} \, dx < \infty \right\}, \]
with the norm

\[ \|u\|_{L^p(x)(\Omega)} = |u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}. \]

Define

\[ W^{1,p}(x)(\Omega) = \left\{ u \in L^p(x)(\Omega) : |\nabla u| \in L^p(x)(\Omega) \right\}, \]
with the norm

\[ \|u\|_{W^{1,p}(x)(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \]

Denote by \( W^{1,p}(0)(\Omega) \), the closure of \( C^\infty_0(\Omega) \) in \( W^{1,p}(x)(\Omega) \).

The basic properties of the spaces \( L^p(x)(\Omega), W^{1,p}(x)(\Omega), \) and \( W^{1,p}(0)(\Omega) \) can be referred to [5-7, 8-11, 13, 17]. Here, we only display some facts which will be used later.

**Proposition 2.1** (see [11]). (i) The spaces \( L^p(x)(\Omega), W^{1,p}(x)(\Omega), \) and \( W^{1,p}(0)(\Omega) \) are separable and reflexive Banach spaces;

(ii) If \( q \in C_\ast(\overline{\Omega}) \) and \( q(x) < p^+(x) \) for any \( x \in (\overline{\Omega}) \), then the embedding from \( W^{1,p}(x)(\Omega) \) to \( L^q(x)(\Omega) \) is compact and continuous;

(iii) There is a constant \( C > 0 \), such that \( |u|_{p(x)} \leq C |\nabla u|_{p(x)}, \forall u \in W^{1,p}(0)(\Omega). \)
By (iii) of Proposition 2.1, we know that $|\nabla u|_{p(x)}$ and $\|u\|$ are equivalent norms on $W^{1,p(x)}(\Omega)$. We will use $|\nabla u|_{p(x)}$ to replace $\|u\|$ in the following discussions.

**Proposition 2.2** (see [10]). Set $\rho(u) = \int_\Omega |u(x)|^{p(x)} \, dx$. For $u, u_k \in L^{p(x)}(\Omega)$, we have

1. For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho\left(\frac{u}{\lambda}\right) = 1$;
2. $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;
3. If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
4. If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
5. $\lim_{k \to \infty} |u_k|_{p(x)} = 0 \Leftrightarrow \lim_{k \to \infty} \rho(u_k) = 0$;
6. $|u_k|_{p(x)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

Similar to Proposition 2.2, we have

**Proposition 2.3.** Set $\phi(u) = \int_\Omega |\nabla u|^{p(x)} + |u(x)|^{p(x)} \, dx$. For $u, u_k \in W^{1,p(x)}(\Omega)$, we have

1. For $u \neq 0$, $\|u\| = \lambda \Leftrightarrow \phi\left(\frac{u}{\lambda}\right) = 1$;
2. $\|u\| < 1 (= 1; > 1) \Leftrightarrow \phi(u) < 1 (= 1; > 1)$;
3. If $\|u\| > 1$, then $\|u\|^{p^-} \leq \phi(u) \leq \|u\|^{p^+}$;
4. If $\|u\| < 1$, then $\|u\|^{p^+} \leq \phi(u) \leq \|u\|^{p^-}$;
5. $\lim_{k \to \infty} \|u_k\| = 0 \Leftrightarrow \lim_{k \to \infty} \phi(u_k) = 0$;
6. $\|u_k\| \to \infty \Leftrightarrow \phi(u_k) \to \infty$. 
Let \( G(u) = \frac{1}{\Omega} \int_{\Omega} |\nabla u|^{p(x)} \, dx, \) \( u \in X. \) We denote \( L = G' : X \to X^*, \) then
\[
(L(u), v) = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx \quad \forall \, u, \, v \in X.
\]

**Proposition 2.4** (see [11]). (i) \( L : X \to X^* \) is a continuous, bounded, and strictly monotone operator;

(ii) \( L \) is a mapping of type \((S_+), i.e., if \( u_n \to u \) in \( X \) and \( \lim_{n \to \infty} (L(u_n) - L(u), u_n - u) \leq 0, \) then \( u_n \to u \) in \( X; \)

(iii) \( L : X \to X^* \) is a homeomorphism.

**Proposition 2.5** (see [19]). Let \( X \) be a reflexive real Banach space; \( I \subseteq \mathbb{R} \) be an interval, \( \Phi : X \to \mathbb{R}, \) a sequentially weakly lower semi-continuous \( C^1 \) functional, whose derivative admits a continuous inverse on \( X^* , \) \( J : X \to \mathbb{R} \) a \( C^1 \) functional with compact derivative. In addition, \( \Phi \) is bounded on each bounded subset of \( X. \) Assume that
\[
\lim_{||x|| \to \infty} (\Phi(x) + \lambda J(x)) = +\infty \quad (2.1)
\]
for all \( \lambda \in I, \) and that there exists \( \rho \in \mathbb{R} \) such that,
\[
\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda (J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda (J(x) + \rho)). \quad (2.2)
\]
Then, there exist a non-empty open set \( A \subseteq I \) and a positive real number \( r \) with the following property: for every \( \lambda \in A \) and every \( C^1 \) functional \( \Psi : X \to \mathbb{R} \) with compact derivative, there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta], \) the equation
\[
\Phi'(x) + \lambda J'(x) + \mu \Psi'(x) = 0
\]
has at least three solutions in \( X, \) whose norms are less than \( r. \)

**Proposition 2.6** (see [18]). Let \( X \) be a non-empty set and \( \Phi, \) \( J \) be two real functionals on \( X. \) Assume that there are \( \gamma > 0, \) \( u_0, \) \( u_1 \in X, \) such that
\[
\Phi(u_0) = J(u_0) = 0, \quad \Phi(u_1) > \gamma,
\]
\[ \sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}. \quad (2.3) \]

Then, for each \( \rho \) satisfies
\[ \sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \rho < \gamma \frac{J(u_1)}{\Phi(u_1)}, \]

one has
\[ \sup_{\lambda \geq 0 \in \Phi^{-1}((-\infty, \gamma])} \inf_{u \in X} (\Phi(u) + \lambda (\rho - J(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda (\rho - J(u))). \]

Let \( E = W^{1, p(x)}(\Omega) \) when \( B = B_1; E = W^{1, p(x)}_0(\Omega) \) when \( B = B_2. \)

And we define for any \( u \in E, \)
\[ \Phi(u) = \int_{\Omega} \frac{1}{p(x)} \left[ |\nabla u|^{p(x)} + |u|^{p(x)} \right] dx, \]
\[ J(u) = \int_{\Omega} F(x, u) dx, \]
\[ \Psi(u) = \int_{\Omega} G(x, u) dx, \]

where \( F(x, t) = \int_{0}^{t} f(x, s) ds, G(x, t) = \int_{0}^{t} g(x, s) ds. \)

In this paper, we will use the following equivalent norm on \( E: \)
\[ \|u\| = \inf \left\{ \lambda > 0 : \int_{\Omega} \lambda \left| \nabla u \right|^{p(x)} dx + \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \]

**Remark 1.** If \( N < \rho^- \leq \rho(x) \) for any \( x \in \overline{\Omega}, \) by Theorem 2.2 in [10], we deduce that \( W^{1, p(x)}(\Omega) \) is continuously embedded in \( W^{1, \rho^-}(\Omega). \) Since \( N < \rho^- \), it follows that \( W^{1, \rho^-}(\Omega) \) is compactly embedded in \( C(\overline{\Omega}). \)

Thus, we deduce that \( W^{1, p(x)}(\Omega) \) is compactly embedded in \( C(\overline{\Omega}). \)

Defining \( \|u\|_x = \sup_{x \in \overline{\Omega}} |u(x)|, \) we find that there exists a positive constant \( c > 0 \) such that
\[ \|v\|_\infty \leq c\|u\|, \quad \forall \ u \in W^{1,p}(\Omega). \]

3. Definition and the Main Result

**Definition 3.1.** We say \( u \in E \) is a weak solution of Equation (P), if
\[
\int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uu) \, dx - \lambda \int_{\Omega} f(x, u) v \, dx - \mu \int_{\Omega} g(x, u) v \, dx = 0,
\]
for any \( v \in E \).

**Theorem 3.2.** Assume that \( \inf_{x \in \Omega} p(x) > N \) for any \( x \in \bar{\Omega} \) and \( f(x, u), g(x, u) \) satisfies \((f_1), (f_2)\). Then there exist an open interval \( \Lambda \subset (0, \infty) \) and two positive real numbers \( \delta, \rho > 0 \) such that each \( \lambda \in \Lambda \) and each \( \mu \in [0, \delta] \), the Equation (P) has at least three solutions whose norms are less than \( \rho \).

**Proof of Theorem 3.2.** Let \( E, \Phi, J, \Psi \) are defined just as above and
\[
I(u) = \int_{\Omega} (|u|^{p(x)} + |\nabla u|^{p(x)}) \, dx.
\]
Then according to the proof of Proposition 3.1 in [16], we obtain that \( \Phi, J, \Psi \in C^1(X, R) \) with the derivatives given by
\[
\langle \Phi'(u), v \rangle = \int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uu) \, dx,
\]
\[
\langle J'(u), v \rangle = -\int_{\Omega} f(x, u) v \, dx,
\]
\[
\langle \Psi'(u), v \rangle = -\int_{\Omega} g(x, u) v \, dx,
\]
for all \( u, v \in X \).

Thus, if there exists \( u \in E \) is a critical point of the operator \( \Phi + \lambda J + \mu \Psi \), that is, to say \( \Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0 \), then we can deduce that
$u \in E$ is a weak solution of Equation (P) by Definition 3.1. Similar to the conclusion of Proposition 2.5, for proving our result, it is enough to verify $\Phi$, $J$, and $\Psi$ satisfy the hypotheses of Proposition 2.5.

It is obvious that $(\Phi')^{-1} : X^* \to X$ exists and continuous since $\Phi' : X \to X^*$ is a homeomorphism by [11]. Moreover, $J', \Psi' : X \to X^*$ is completely continuous because of assumption $(f_1)$ from [11], which implies $J', \Psi'$ is compact.

Next, we will verify that condition (2.1) in Proposition 2.5 is fulfilled. In fact, by relation (3) in Proposition 2.3, we have

$$\Phi(u) \geq \frac{1}{p^*} \int_{\Omega} \left( |\nabla u|^{p(x)} + |u|^{\alpha(x)} \right) dx = \frac{1}{p^*} I(u) \geq \frac{1}{p^*} ||u||^{p^-}, \quad \forall \, u \in X, \, ||u|| > 1.$$ 

On the other hand, due to assumption $(f_1)$, we have

$$J(u) = -\int_{\Omega} F(x, u) dx = \int_{\Omega} - F(x, u) dx,$$

and

$$|F(x, t)| \leq c_1 |t| + c_2 \frac{1}{\alpha(x)} |t|^\alpha(x),$$

therefore,

$$J(u) \geq -c_1 \int_{\Omega} |u| dx - c_2 \int_{\Omega} \frac{1}{\alpha(x)} |u|^\alpha(x)$$

$$\geq -c_3 ||u|| - c_2 \cdot \frac{1}{\alpha^+} \cdot \left( \int_{\Omega} (|u|^\alpha^+ + |u|^\alpha^-) dx \right)$$

$$= -c_3 ||u|| - c_4 (|u|^\alpha^+ + |u|^\alpha^-)$$

$$\geq -c_3 ||u|| - c_5 (||u||^\alpha^+ + ||u||^\alpha^-)$$

here we used Proposition 2.2. It follows that
\[
\Phi(u) + \lambda J(u) \geq \frac{1}{p^*} \|u\|^{p^*} - \lambda c_3 \|u\| - \lambda c_5 (\|u\|^\alpha + \|u\|^\alpha^+), \quad \forall u \in E.
\]

Since \(1 < \alpha^+ < p^*\), it follows that \(\lim_{\|u\| \to x} (\Phi(u) + \lambda J(u)) = \infty\) and (2.1) is verified.

In the sequel, we verify that conditions (2.2) in the Proposition 2.5 is satisfied. It suffices to verify the conditions of Proposition 2.6.

By \(F'(x, t) = f(x, t)\) and assumption \((f_2)\), \(F(x, t)\) is increasing for \(t \in (t_0, \infty), \ t_0 > 1 \) and decreasing for \(t \in (0, 1)\), uniformly with respect to \(x\) and \(F(x, 0) = 0\) is obvious. Because \(F(x, t) \to \infty\) when \(t \to \infty\) by \((f_2)\). Then, there exists a real number \(\delta > t_0\) such that

\[
F(x, t) \geq 0 = F(x, 0) \geq F(x, \tau), \quad \forall \ x \in \Omega, \ t > \delta, \ \tau \in (0, 1).
\]

Because \(\Omega \subset \mathbb{R}^N\) is bounded, there exists a cube \(Q\) such that \(\Omega \subset Q\). Suppose the length of \(Q\) is \(L > 0\). Divide \(L\) into \(K\) equal parts, then \(Q\) is divided into \(K^N\) subcubes and \(s = \frac{L}{K}\) is the length of small cubes. Note the number of small cubes which lie in \(\Omega\) entirely is \(S(K)\). It is obvious that \(S(K) \to \infty\) when \(K \to \infty\). We denote \(Q(x_0, s)\) is a cube with center \(x_0\) and its length is \(s\), moreover, \(Q(x_0, s) \subset \Omega\).

Let \(a, b\) be two real numbers such that \(0 < a < \min\{1, c\}\) with \(c\) given in Remark 1 and \(b > \delta\). When \(t \in [0, a]\), we have \(F(x, t) \leq F(x, 0)\), it follows that

\[
\int_{\Omega} \sup_{0 \leq t \leq a} F(x, t) \ dx \leq \int_{\Omega} F(x, 0) \ dx = 0.
\]

We define

\[
v_i(x) = \begin{cases} 
    b, & \text{if } x \in Q_i(x_i, \frac{s}{2}), \\
    0, & \text{if } x \in Q_i(x_i, s) \setminus Q_i(x_i, \frac{s}{2} + \varepsilon),
\end{cases}
\]
in $Q_i = Q_i(x_i, s)$, $1 \leq i \leq S(K)$, $v_i(x) \in C^\infty_0(Q_i, [0, b])$ for $\varepsilon > 0$ and $\varepsilon$ will be determined later.

Consider $u_0, u_1 \in X, u_0(x) = 0$ for any $x \in \Omega$ and $u_1(x) =
\begin{cases}
v_i, & \text{if } x \in Q_i, \\
0, & \text{if } x \in \Omega \setminus \bigcup_{i=1}^{S(K)} Q_i.
\end{cases}

We also define $r = \frac{1}{p^+} \cdot \left( \frac{a}{\varepsilon} \right)^{p^+}$, clearly, $r \in (0, 1)$. A simple computation implies

$$\Phi(u_0) = J(u_0) = 0,$$

and

$$\Phi(u_1) = \int_{\Omega} \frac{1}{p(x)} \left( |\nabla u_1(x)|^{p(x)} + |u_1(x)|^{p(x)} \right) dx$$

\begin{align*}
&\geq \frac{1}{p^+} \sum_{i=1}^{S(K)} \int_{Q_i} \left( |\nabla v_i(x)|^{p(x)} + |v_i(x)|^{p(x)} \right) dx \\
&= \frac{1}{p^+} \sum_{i=1}^{S(K)} \left[ \int_{Q_i(x_i, \frac{s}{2})} b^{p(x)} dx \right. \\
&\quad + \left. \int_{Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})} \left( |\nabla v_i(x)|^{p(x)} + |v_i(x)|^{p(x)} \right) dx \right] \\
&\geq \frac{1}{p^+} \sum_{i=1}^{S(K)} \int_{Q_i(x_i, \frac{s}{2})} b^{p(x)} dx \\
&\geq \frac{1}{p^+} \sum_{i=1}^{S(K)} \left( b^{p^+} \cdot \left( \frac{s}{2} \right)^{N} \right) \\
&= \frac{1}{p^+} \cdot S(K) \cdot b^{p^+} \cdot \left( \frac{s}{2} \right)^{N}.
\end{align*}
Next, we can choose $K$ large enough such that $S(K) > \frac{r \cdot P^+}{b^0 - (\frac{s}{2})^N}$, so

$\Phi(u_1) > r$.

Moreover,

$$- J(u_1) = \int_{\Omega} F(x, u_1(x)) \, dx$$

$$= \int_{\bigcup_{i=1}^{S(K)} Q_i} F(x, u_1(x)) \, dx$$

$$= \sum_{i=1}^{S(K)} \int_{Q_i} F(x, v_i(x)) \, dx$$

$$= \sum_{i=1}^{S(K)} \left( \int_{Q_i(x_i, \frac{s}{2})} F(x, b) \, dx + \int_{Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})} F(x, v_i) \, dx \right).$$

Furthermore, $F(x, b) > 0$ and $b > \delta$, we let $\int_{Q_i(x_i, \frac{s}{2})} F(x, b) \, dx = b_i > 0$.

There exists a constant $M > 0$ such that $|F(x, u_i)| \leq M$ in $Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})$, since $0 < u_i(x) < b$ in $Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})$, it follows that

$$\int_{Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2})} F(x, u_i) \, dx \geq -M \cdot O(\varepsilon),$$

since

$$\left| Q_i(x_i, \frac{s}{2} + \varepsilon) \setminus Q_i(x_i, \frac{s}{2}) \right| = \left( \frac{s}{2} + \varepsilon \right)^N - \left( \frac{s}{2} \right)^N = O(\varepsilon),$$

so

$$- J(u_1) \geq \sum_{i=1}^{S(K)} (b_i - M \cdot O(\varepsilon)) = \sum_{i=1}^{S(K)} b_i - S(K) \cdot M \cdot O(\varepsilon).$$
We can choose \( \varepsilon > 0 \) small enough such that \( S(K) \cdot M \cdot O(\varepsilon) < \frac{1}{2} \sum_{i=1}^{S(K)} b_i \), it follows that
\[
-J(u_1) > \frac{1}{2} \sum_{i=1}^{S(K)} b_i > 0.
\]

Then, we obtain
\[
\int_{\Omega} \sup_{0 \leq t \leq a} F(x, t) \, dx \leq 0 < r \cdot \frac{-J(u_1)}{\Phi(u_1)}.
\]

Next, we consider the case when \( u \in X \) with \( \Phi(u) \leq r < 1 \). Since \( \frac{1}{p^+} I(u) \leq \Phi(u) \leq r \), we obtain \( I(u) \leq p^+ \cdot r = (\frac{a}{c})^{p^+} < 1 \), it follows that \( \|u\| < 1 \) by (4). Furthermore, by (6), it is clear that
\[
\frac{1}{p^+} \cdot \|u\|^{p^+} \leq \frac{1}{p^+} \cdot I(u) \leq \Phi(u) \leq r.
\]

Thus, using Remark 1, for any \( u \in X \) with \( \Phi(u) \leq r \), we obtain
\[
\|u(x)\| \leq c \cdot \|u\| \leq c \cdot (p^+ \cdot r)^{1/p^+} = a, \quad \forall \ x \in \Omega.
\]

The above inequality shows
\[
\Phi^{-1}((-\infty, r]) \subset [0, a].
\]

It follows that
\[
\sup_{u \in \Phi^{-1}((-\infty, r])} -J(u) \leq \int_{\Omega} \sup_{0 \leq t \leq a} F(x, t) \, dx < r \cdot \frac{-J(u_1)}{\Phi(u_1)}.
\]

Set \( I = [0, +\infty) \), moreover, \( \Phi(u) \) and \( -J(u) \) satisfy the assumption of Proposition 2.6, so using the Proposition 2.6, we can easily obtain that (2.2) is satisfied.

Thus, \( \Phi, J, \) and \( \Psi \) satisfy all the assumptions of Proposition 2.5, and the proof is complete.
References


